The Cayley isomorphism property for groups of order $8p$

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January 18, 2013

Abstract

For every prime $p > 3$ we prove that $Q \times \mathbb{Z}_p$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ are DCI-groups. This result completes the description of CI-groups of order $8p$.

1 Introduction

Let G be a finite group and S a subset of G. The Cayley graph $Cay(G, S)$ is defined by having the vertex set G and g is adjacent to h if and only if $gh^{-1} \in S$. The set S is called the connection set of the Cayley graph $Cay(G, S)$. A Cayley graph $Cay(G, S)$ is undirected if and only if $S = S^{-1}$, where $S^{-1} = \{ s^{-1} \in G \mid s \in S \}$. Every right multiplication via elements of G is an automorphism of $Cay(G, S)$, so the automorphism group of every Cayley graph on G contains a regular subgroup isomorphic to G. Moreover, this property characterises the Cayley graphs of G.

It is clear that $Cay(G, S) \cong Cay(G, S^\mu)$ for every $\mu \in Aut(G)$. A Cayley graph $Cay(G, S)$ is said to be a CI-graph if, for each $T \subset G$, the Cayley graphs $Cay(G, S)$ and $Cay(G, T)$ are isomorphic if and only if there is an automorphism μ of G such that $S^\mu = T$. Furthermore, a group G is called a DCI-group if every Cayley graph of G is a CI-graph and it is called a CI-group if every undirected Cayley graph of G is a CI-graph.

It was proved in [5] that $\langle a, z \mid a^p = 1, z^8 = 1, z^{-1}az = a^{-1} \rangle$ is a CI-group, though not a DCI-group. Let G be a DCI-group of order $8p$, where p is odd prime. It can easily be seen that every subgroup of a DCI-group is also a DCI-group. It follows that the Sylow 2-subgroup of G can only be the quaternion group Q of order 8 or $\mathbb{Z}_2^3$.

*Research supported by the Hungarian Scientific Fund (OTKA), grant no. K84233

If $p > 8$ or $p = 5$, then by Sylow's Theorem the Sylow p-subgroup of G is a normal subgroup therefore G is isomorphic to one of the following groups: $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, $Q \times \mathbb{Z}_p$, $\mathbb{Z}_2^3 \ltimes \mathbb{Z}_p$ or $Q \ltimes \mathbb{Z}_p$. It was proved in [2] in that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to ternary relational structures if $p \geq 11$. Moreover, Dobson and Spiga proved in [3] that $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a CI-group with respect to color binary relational structures for all primes p and it is a CI-group with respect to color ternary relational structures if and only if $p \neq 3$ and 7.

Spiga proved in [6] that $Q \times \mathbb{Z}_3$ is not a CI-group with respect to colour ternary relational structures and the non-nilpotent group $Q \rtimes \mathbb{Z}_3$ is not a CI-group.

If $p = 7$, then either the Sylow 7-subgroup is normal, in which case G is as before, or G has 8 Sylow 7-subgroups, when $G \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_7$. The non-nilpotent groups above are not DCI-groups, see [4]. We show that the other groups are DCI-groups.

Theorem 1. *For every prime $p > 3$ the groups $Q \times \mathbb{Z}_p$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ are DCI-groups.*

Our paper is organized as follows. In section 2 we introduce the notation that will be used throughout this paper. In section 3 we collect important ideas that we will use in the proof of Theorem 1. Section 4 contains the proof of Theorem 1 for primes $p > 8$ and Section 5 contains the proof of Theorem 1 for $p = 5$ and 7.

2 Technical details

In this section we introduce some notation. Let G be a group. We use $H \leq G$ to denote that H is a subgroup of G and by $N_G(H)$ and $C_G(H)$ we denote the normalizer and the centralizer of H in G, respectively.

Let us assume that the group H acts on the set Ω and let G be an arbitrary group. Then by $G \wr_\Omega H$ we denote the wreath product of G and H. Every element $g \in G \wr_\Omega H$ can be uniquely written as hk, where $k \in K = \prod_{\omega\in\Omega} G_\omega$ and $h \in H$. The group $K = \prod_{\omega\in\Omega} G_\omega$ is called the base group of $G \wr_\Omega H$ and the elements of K can be treated as functions from Ω to G. If $g \in G \wr_\Omega H$ and $g = hk$ we denote k by $(g)_b$. In order to simplify the notation Ω will be omitted if it is clear from the definition of H and we will write $G \wr H$.

The symmetric group on the set Ω will be denoted by $Sym(\Omega)$. Let G be a permutation group on the set Ω. For a G-invariant partition $\mathcal{B}$ of the set Ω we use $G^{\mathcal{B}}$ to denote the permutation group on $\mathcal{B}$ induced by the action of G and similarly, for every $g \in G$ we denote by $g^{\mathcal{B}}$ the action of g on the partition $\mathcal{B}$.

For a group G, let $\hat{G}$ denote the subgroup of the symmetric group $Sym(G)$ formed by the elements of G acting by right multiplication on G. For every Cayley graph $\Gamma = Cay(G, S)$ the subgroup $\hat{G}$ of $Sym(G)$ is contained in $Aut(\Gamma)$.

Definition 1. *Let $G \leq Sym(\Omega)$ be a permutation group. Let*

$$G^{(2)} = \left\{ \pi \in Sym(\Omega) \middle| \begin{array}{ll} \forall a, b \in \Omega\ \exists g_{a,b} \in G \ \textit{with} & \pi(a) = g_{a,b}(a) \ \textit{and} \\ & \pi(b) = g_{a,b}(b) \end{array} \right\}.$$

We say that $G^{(2)}$ is the 2-closure of the permutation group G.

Lemma 1. *Let Γ be a graph. If $G \leq Aut(\Gamma)$, then $G^{(2)} \leq Aut(\Gamma)$.*

3 Basic ideas

In this section we collect some results and some important ideas that we will use in the proof of Theorem 1.

We begin with a fundamental lemma that we will use all along this paper.

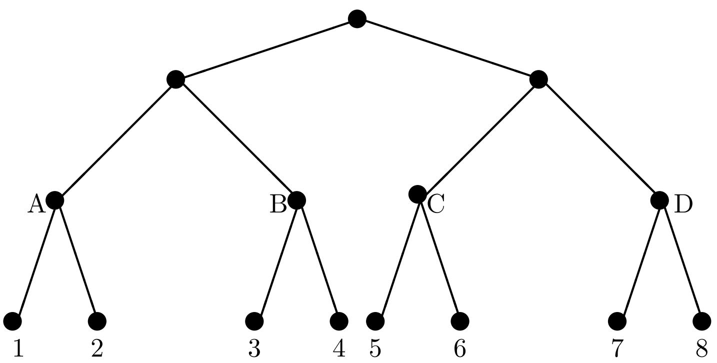
Lemma 2 (Babai [1])**.** *$Cay(G, S)$ is a CI-graph if and only if for every regular subgroup $\mathring{G}$ of $Aut(Cay(G, S))$ isomorphic to G there is a $\mu \in Aut(Cay(G, S))$ such that $\mathring{G}^{\mu} = \hat{G}$.*

We introduce the following definition.

Definition 2. *(a) We say that a Cayley graph $Cay(G, S)$ is a $CI^{(2)}$-graph iff for every regular subgroup $\mathring{G}$ of $Aut(Cay(G, S))$ isomorphic to G there is a $\sigma \in \langle \mathring{G}, \hat{G} \rangle^{(2)}$ such that $\mathring{G}^{\sigma} = \hat{G}$.*

(b) A group G is called a $DCI^{(2)}$-group if for every $S \subset G$ the Cayley graph $Cay(G, S)$ is a $CI^{(2)}$-graph.

Let us assume that $A = Aut(Cay(G, S)) \leq Sym(8p)$ contains two copies of regular subgroups, $\mathring{Q} \times \mathring{\mathbb{Z}}_p$ and $\hat{Q} \times \hat{\mathbb{Z}}_p$. By Sylow's theorem we may assume that $\mathring{\mathbb{Z}}_p$ and $\hat{\mathbb{Z}}_p$ are in the same Sylow p-subgroup P of $Sym(8p)$. If $p > 8$, then P is isomorphic to $\mathbb{Z}_p^8$. Moreover, P is generated by 8 disjoint p-cycles. It follows that both $\mathring{Q}$ and $\hat{Q}$ normalize P so we may assume that $\mathring{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of $N_A(P)$. Let P_2 denote a Sylow 2-subgroup of $Sym(8)$. It is also well known that P_2 is isomorphic to the automorphism group of the following graph Δ:



Every automorphism of Δ permutes the leaves of the graph and the permutation of the leaves determines the automorphism, therefore $Aut(\Delta)$ can naturally be embedded into $Sym(8)$.

It is easy to see that the same holds if we change $Q \times \mathbb{Z}_p$ to $\mathbb{Z}_2^3 \times \mathbb{Z}_p$.

Lemma 3. *(a) There are exactly two regular subgroups of P_2 which are isomorphic to Q.*

(b) There are exactly two regular subgroups of P_2 which are isomorphic to $\mathbb{Z}_2^3$.

Proof. (a) Let Q be a regular subgroup of $Aut(\Delta)$ isomorphic to the quaternion group with generators i and j. For every $1 \leq m \leq 4$ there is a $q_m \in Q$ such that $q_m(2m-1) = 2m$. These are automorphisms of Δ so $q_m(2m) = 2m-1$ and hence the order of q_m is 2. There is only one involution in Q so $q_m = i^2$ for every $1 \leq m \leq 4$ and this fact determines completely the action of i^2 on Δ.

We can assume that $i(1) = 3$. Such an isomorphism of Δ fixes setwise $\{1, 2, 3, 4\}$ so we have that $i(3) = 2$, $i(2) = 4$ and $i(4) = 1$ since i is of order 4. Using again the fact that Q is regular on Δ and $i^2(5) = 6$, we get that there are two choices for the action of i: $i = (1324)(5768)$ or $i = (1324)(5867)$.

We can also assume that $j(1) = 5$. This implies that $j(5) = j^2(1) = i^2(1) = 2$, and $j(2) = 6$ since $j \in Aut(\Delta)$ and $j(6) = 1$. The action of i determines the action of j on Δ since $iji = j$. Applying this to the leaf 3 we get that $j(3) = 8$ if $i = (1324)(5768)$ and $j(3) = 7$ if $i = (1324)(5867)$ so there is no more choice for the action of j. Finally, i and j generate Q and this gives the result.

(b) Let us assume that $x \in \mathbb{Z}_2^3$ such that $x(1) = 2$ A fixed point free automorphism of Γ of order 2 which maps 1 to 2 will map 3 to 4. There is an $y \in \mathbb{Z}_2^3$ such that $y(1) = 5$. Such an automorphism of Γ maps 2 to 6 so we have that $x(5) = 6$ since x and y commute. This determines x completely so we have that $x =$(12)(34)(56)(78).

We have two possibilities for $y(3)$. If $y(3) = 7$, then $y = (15)(26)(37)(48)$ and if $y(3) = 8$, then $y = (15)(26)(38)(47)$. The third generator of the group $\mathbb{Z}_2^3$ which maps 1 to 3 is determined by x and y since $\mathbb{Z}_2^3$ is abelian. ■

The previous proof also gives the following.

Lemma 4. *(a) The following two pairs of permutations generate the two regular subgroups of $Aut(\Delta) \leq Sym(8)$ isomorphic to Q:*

$$i_1 = (1324)(5768),\ j_1 = (1526)(3748)$$

and

$$i_2 = (1324)(5867),\ j_2 = (1526)(3847)$$

(b) *The elements of these regular subgroups of $Aut(\Delta)$ are the following:*

Q_l :	Q_r :
id	id
(12)(34)(56)(78)	(12)(34)(56)(78)
(1324)(5768)	(1324)(5867)
(1423)(5867)	(1423)(5768)
(1526)(3748)	(1526)(3847)
(1625)(3847)	(1625)(3748)
(1728)(3546)	(1728)(3645)
(1827)(3645)	(1827)(3546)

Using the following identification Q_l and Q_r act on Q by left-multiplication and right-multiplication, respectively:

1	2	3	4	5	6	7	8
1	-1	i	$-i$	j	$-j$	k	$-k$.

(c) *The following permutations generate two regular subgroups of $Aut(\Delta) \leq Sym(8)$ isomorphic to $\mathbb{Z}_2^3$.*
A_1 is generated by:

$$x_1 = (12)(34)(56)(78),\ x_2 = (13)(24)(57)(68),\ x_3 = (15)(26)(37)(48)$$

and A_2 is generated by:

$$y_1 = (12)(34)(56)(78),\ y_2 = (13)(24)(58)(67),\ y_3 = (15)(26)(38)(47).$$

Lemma 5. *Let us assume that $G_1 \leq P_2$ is generated by two different regular subgroups Q_a and Q_b of $Aut(\Delta)$ which are isomorphic to Q and $G_2 \leq P_2$ is generated by two different regular subgroups A_1 and A_2 of $Aut(\Delta)$ which are isomorphic to $\mathbb{Z}_2^3$. Then $G_1 = G_2$.*

Proof. It is clear that $|P_2| = |Aut(\Delta)| = 2^7$. One can see using Lemma 4 (a) that G_1 and G_2 are generated by even permutations. Both G_1 and G_2 induce an action on the set $V = \{A,\ B,\ C,\ D\}$ which is a set of vertices of Δ and it is easy to verify that every permutation of V induced by G_1 and G_2 is even. This shows that G_1 and G_2 are contained in a subgroup of P_2 of cardinality 2^5.

Lemma 4 (b) shows that $|Q_a \cap Q_b| = 2$ and one can also check that $|A_1 \cap A_2| = 2$. This gives $|G_1| \geq 2^5$ and $|G_2| \geq 2^5$, finishing the proof of Lemma 5. ■

Proposition 1. *(a) The quaternion group Q is a $DCI^{(2)}$-group.*

(b) $\mathbb{Z}_2^3$ is a $DCI^{(2)}$-group.

Proof. (a) Let Q_a and Q_b be two regular subgroups of $Sym(8)$ isomorphic to the quaternion group Q. By Sylow's theorem we may assume that Q_a and Q_b lie in the same Sylow 2-subgroup of $H = \langle Q_a, Q_b \rangle$. Since every Sylow

2-subgroup of H is contained in a Sylow 2-subgroup of $Sym(8)$, we may assume that Q_a and Q_b are subgroups of $Aut(\Delta)$.

Our aim is to find an element $\pi \in \langle Q_a, Q_b\rangle^{(2)}$ such that $Q_a^\pi = Q_b$ so let us assume that $Q_a \neq Q_b$. Using Lemma 4 (a) we may also assume that Q_a and Q_b are generated by the permutations $(1324)(5768), (1526)(3748)$ and $(1324)(5867), (1526)(3847)$, respectively. Lemma 4 (b) shows that H contains the following three permutations:

$$(12)(34) = (1324)(5768)(1324)(5867)$$
$$(12)(56) = (1526)(3748)(1526)(3847)$$
$$(12)(78) = (1728)(3546)(1728)(3645).$$

Now one can easily see that the permutation (12) is in $H^{(2)}$. Finally, it is also easy to check using Lemma 4 (b) that $Q_a^{(12)} = Q_b$.

(b) Let A_1 and A_2 be two regular subgroups of $Sym(8)$ isomorphic to $\mathbb{Z}_2^3$. Let H' denote the group generated by A_1 and A_2. Similarly to the previous case we may assume that A_1 and A_2 are different regular subgroups of $Aut(\Delta)$. By Lemma 4 A_1 and A_2 are generated by the permutations $x_1 = (12)(34)(56)(78)$, $x_2 = (13)(24)(57)(68)$, $x_3 = (15)(26)(37)(48)$ and $x_1 = y_1 = (12)(34)(56)(78)$, $y_2 = (13)(24)(58)(67)$, $y_3 = (15)(26)(38)(47)$, respectively.

By Lemma 5 the group H' contains the permutations $(12)(34)$, $(12)(56)$ and $(12)(78)$. Therefore H' contains the permutation (12) which conjugates A_1 to A_2 since (12) centralizes x_1 and we also have $(12)x_2(12) = y_2y_1$ and $(12)x_3(12) = y_1y_3$, finishing the proof of Proposition 1. ■

Definition 3. *Let Γ be an arbitrary graph and $A, B \subset V(\Gamma)$ such that $A \cap B = \emptyset$. We write $A \sim B$ if one of the following four possibilities holds:*

(a) For every $a \in A$ and $b \in B$ there is an edge from a to b but there is no edge from b to a.

(b) For every $a \in A$ and $b \in B$ there is an edge from b to a but there is no edge from a to b.

(c) For every $a \in A$ and $b \in B$ the vertices a and b are connected with an undirected edge.

(d) There is no edge between A and B.

We also write $A \nsim B$ if none of the previous four possibilities holds.

Lemma 6. *Let A, B be two disjoint subsets of cardinality p of a graph. We write $A \cup B = \mathbb{Z}_p \cup \mathbb{Z}_p$. Let us assume that $\hat{\mathbb{Z}}_p$ acts naturally on $A \cup B$ and for a generator $\mathring{a}$ of the cyclic group $\mathring{Z}_p$ the action of $\mathring{a}$ is defined by $\mathring{a}(a_1, a_2) = (a_1 + b, a_2 + c)$ for some $b, c \in \mathbb{Z}_p$.*

(a) *If $b = c$, then the action of $\mathbb{Z}_p$ and $\mathring{\mathbb{Z}}_p$ on $A \cup B$ are the same.*

(b) *If $A \nsim B$, then $b = c$.*

(c) *If $A \sim B$, then every $\pi \in Sym(A \cup B)$ which fixes A and B setwise is an automorphism of the graph defined on $A \cup B$ if $\pi \restriction A \in Aut(A)$ and $\pi \restriction B \in Aut(B)$.*

Proof. These statements are obvious. ■

4 Main result for $p > 8$

In this section we will prove that $Q \times \mathbb{Z}_p$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ are DCI-groups if $p > 8$. We will first prove it for $Q \times \mathbb{Z}_p$ and then we will repeat the argument for the case of $\mathbb{Z}_2^3 \times \mathbb{Z}_p$.

Proposition 2. *For every prime $p > 8$, the group $Q \times \mathbb{Z}_p$ is a DCI-group.*

Our technique is based on Lemma 2 so we have to fix a Cayley graph $\Gamma = Cay(Q \times \mathbb{Z}_p, S)$. Let $A = Aut(\Gamma)$ and $\mathring{G} = \mathring{Q} \times \mathring{\mathbb{Z}}_p$ be a regular subgroup of A isomorphic to $Q \times \mathbb{Z}_p$. In order to prove Proposition 2 we have to find an $\alpha \in A$ such that $\mathring{G}^\alpha = \hat{G} = \hat{Q} \times \hat{\mathbb{Z}}_p$ what we will achieve in three steps.

4.1 Step 1

We may assume $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$ lie in the same Sylow p-subgroup P of $Sym(8p)$. Then both $\mathring{Q}$ and $\hat{Q}$ are subgroups of $N_{Sym(8p)}(P) \cap A$ so we may assume that $\mathring{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of $N_{Sym(8p)}(P) \cap A$ which is contained in a Sylow 2-subgroup of A.

The Sylow p-subgroup P gives a partition $\mathcal{B} = \{B_1, B_2, \ldots, B_8\}$ of the vertices of Γ, where $|B_i| = p$ for every $i = 1, \ldots, 8$ and $\mathcal{B}$ is P-invariant. It is easy to see that $\mathcal{B}$ is invariant under the action of $\hat{Q}$ and $\mathring{Q}$ and hence $\langle \hat{G}, \mathring{G} \rangle \leq Sym(p) \wr Sym(8)$. Moreover, both $\mathring{G}$ and $\hat{G}$ are regular so $\mathring{Q}$ and $\hat{Q}$ induce regular action on $\mathcal{B}$ which we denote by Q_1 and Q_2, respectively. The assumption that $\mathring{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of A implies that Q_1 and Q_2 are in the same Sylow 2-subgroup of $Sym(8)$.

4.2 Step 2

Let us assume that $Q_1 \neq Q_2$. We intend to find an element $\alpha \in A$ such that $\left(\mathring{Q}^\alpha\right)^{\mathcal{B}} = Q_2$.

Using Lemma 4(b) we can assume that $\mathring{Q}$ is generated by the permutations $\mathring{i}$ and $\mathring{j}$ such that $\mathring{i}$ and $\mathring{j}$ induce the permutations $(B_1B_3B_2B_4)(B_5B_7B_6B_8)$ and $(B_1B_5B_2B_6)(B_3B_7B_4B_8)$, respectively. Similarly, $\hat{Q}$ is generated by $\hat{i}$ and $\hat{j}$ with $\hat{i}^{\mathcal{B}} = (B_1B_3B_2B_4)(B_5B_8B_6B_7)$ and $\hat{j}^{\mathcal{B}} = (B_1B_5B_2B_6)(B_3B_8B_4B_7)$.

We define a graph Γ_0 on $\mathcal{B}$ such that B_i is connected to B_j if and only if $B_i \not\sim B_j$. This is an undirected graph with vertex set $\mathcal{B}$ and both Q_1 and Q_2 are regular subgroups of $Aut(\Gamma_0)$. It follows that Γ_0 is a Cayley graph of the quaternion group of order 8.

Definition 4. *(a) For a pair $(B_i, B_j) \in \mathcal{B}^2$ we write $B_i \equiv B_j$ if either there exists a path $C_1, C_2, \ldots, C_n$ in Γ_0 such that $C_1 = B_1$, $C_n = B_2$ or $i = j$.*

(b) For a pair $(B_i, B_j) \in \mathcal{B}^2$ we write $B_i \not\equiv B_j$ if $B_i \equiv B_j$ does not hold.

(c) If both H and K are subsets of the vertices of Γ_0 such that $H \cap K = \emptyset$ and for every $B_i \in H$, $B_j \in K$ we have $B_i \not\equiv B_j$, then we write $H \not\equiv K$.

Observation 1. *(a) The relation $\equiv$ defines an equivalence relation on $\mathcal{B}$. The equivalence classes defined by the relation $\equiv$ will be called equivalence classes.*

(b) Since Q_1 acts transitively on $\mathcal{B}$ we have that the size of the equivalence classes defined by the relation $\equiv$ divides 8.

We can also define a colored graph Γ_1 on $\mathcal{B}$ by coloring the edges of the complete directed graph on 8 points. B_i is connected to B_j with the same color as B'_i is connected to B'_j in Γ_1 if and only if there exists a graph isomorphism ϕ from $B_i \cup B_j$ to $B'_i \cup B'_j$ such that $\phi(B_i) = B'_i$ and $\phi(B_j) = B'_j$. The graph Γ_1 is a colored Cayley graph of the quaternion group. Moreover, both Q_1 and Q_2 act regularly on Γ_1. Using the fact that Q has property $DCI^{(2)}$ it is clear that there exists an $\alpha' \in \langle Q_1, Q_2 \rangle^{(2)} \le Aut(\Gamma_1)$ such that $Q_2^{\alpha'} = Q_1$. We would like to lift α' to an automorphism α of Γ such that $\alpha^{\mathcal{B}} = \alpha'$.

(a) Let us assume first that Γ_0 is a connected graph.

Lemma 7. *(a) $\mathring{Q} \times \mathring{\mathbb{Z}}_p \le \hat{\mathbb{Z}}_p \wr Sym(8)$.*

(b) If $\mathring{Q} \times \mathring{\mathbb{Z}}_p \le \hat{\mathbb{Z}}_p \wr Sym(8)$, then for every $\mathring{q} \in \mathring{Q}$ we have $(\mathring{q})_b = id$.

Proof. (a) We first prove that $\hat{\mathbb{Z}}_p = \mathring{\mathbb{Z}}_p$. Let x and y generate $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$, respectively. We can assume that $x \restriction B_1 = y \restriction B_1$. Using Lemma 6(b) we get that $x \restriction B_i = y \restriction B_i$ if there exists a path in Γ_0 from B_1 to B_i. This shows that $x = y$ since Γ_0 is connected. Moreover, $\mathring{Q} \times \mathbb{Z}_p \le \hat{\mathbb{Z}}_p \wr Sym(8)$ since the elements of $\mathring{\mathbb{Z}}_p$ and the elements of $\mathring{Q}$ commute.

(b) Let $A' = A \cap \hat{\mathbb{Z}}_p \wr Sym(8)$. We have already assumed that $\mathring{Q}$ and $\hat{Q}$ lie in the same Sylow 2-subgroup of A'. Let $\mathring{q}$ be an arbitrary element of $\mathring{Q}$. For every $(a, u) \in Q \times \mathbb{Z}_p$ we have $\mathring{q}(a, u) = (b, u+t)$ for some $b \in Q$ and $t \in \mathbb{Z}_p$, where t only depends on $\mathring{q}$ and a since $\mathring{q} \le \hat{\mathbb{Z}}_p \wr Sym(8)$. The permutation group $\hat{G}$ is transitive, hence there exist $\hat{q}_1, \hat{q}_2 \in \hat{Q}$ such that $\hat{q}_1(1, u) = (a, u)$ and $\hat{q}_2(b, u + t) = (1, u + t)$. The order of $\hat{q}_2 \mathring{q} \hat{q}_1$ is a power of 2 since $\hat{q}_2, \mathring{q}, \hat{q}_1$ lie in a Sylow 2-subgroup. Therefore $t = 0$ and hence $(\mathring{q})_b = id$.

■

Lemma 7 says that if Γ_0 is connected, then $\langle \mathring{Q}, \hat{Q} \rangle \leq \hat{Z}_p \wr Sym(8)$ and $(q)_b = id$ for every $q \in \langle \mathring{Q}, \hat{Q} \rangle$. Therefore we can define $\alpha = \alpha' id_{\mathcal{B}}$ to be an element of the wreath product $\hat{Z}_p \wr Sym(8)$ and clearly $\alpha' id_{\mathcal{B}}$ is an element of A with $\alpha^{\mathcal{B}} = \alpha'$.

(b) Let us assume that Γ_0 is the empty graph.

Then Lemma 6(c) shows that every permutation in $\langle Q_1, Q_2 \rangle^{(2)}$ lifts to an automorphism of Γ.

(c) Let us assume that Γ_0 is neither connected nor the empty graph.

Observation 2. *If $Q_1 \neq Q_2$, then $\langle \mathring{Q}, \hat{Q} \rangle \leq A$ contains β_1, β_2, β_3 such that*

$$\beta_1^{\mathcal{B}} = (B_1B_2)(B_3B_4),\ \beta_2^{\mathcal{B}} = (B_1B_2)(B_5B_6),\ \beta_3^{\mathcal{B}} = (B_1B_2)(B_7B_8).$$

Proof. By Lemma 4 the elements β_1, β_2, β_3 can be generated as products of an element of $\mathring{Q}$ and $\hat{Q}$. ■

Lemma 8. *We claim that $B_{2k-1} \equiv B_{2k}$ for $k = 1, 2, 3, 4$.*

Proof. Since Γ_0 is a Cayley graph and Q_1 is transitive on the pairs of the form (B_{2k-1}, B_{2k}) it is enough to prove that $B_1 \equiv B_2$. If $B_1 \nsim B_2$, then $B_1 \equiv B_2$ so we can assume that $B_1 \sim B_2$. Since Γ_0 is not the empty graph B_1 is connected to B_l for some $l > 2$. By Observation (2) there exists $\beta \in A$ such that $\beta(B_1) = B_2$ and $\beta(B_l) = B_l$. This shows that $B_2 \nsim B_l$ and hence $B_1 \equiv B_2$. ■

Γ_0 is not connected so the size of the equivalence classes defined by $\equiv$ cannot be bigger than 4. Since $B_1 \equiv B_2$ there exists a partition $H_1 \cup H_2 = \mathcal{B}$ such that $|H_1| = |H_2| = 4$, $B_1, B_2 \in H_1$ and $H_1 \not\equiv H_2$.

Lemma 9. *There exists $\alpha \in A$ such that $\alpha^{\mathcal{B}} = \alpha'$.*

Proof. Let us assume first that $H_1 = \{B_1, B_2, B_3, B_4\}$. Then we define α_1 to be equal to β_2 on H_1 and the identity on H_2. Using Lemma 6(b) we get that α_1 is in $\langle \mathring{Q}, \hat{Q} \rangle^{(2)}$.

If $H_1 = \{B_1, B_2, B_5, B_6\}$ or $H_1 = \{B_1, B_2, B_7, B_8\}$, then we define α_2 by $\alpha_2 \restriction H_1 = \beta_1$ and $\alpha_2 \restriction H_2 = id$. Lemma 6(b) shows again that $\alpha_2 \in A$.

It is easy to see that $\alpha_1^{\mathcal{B}} = \alpha_2^{\mathcal{B}} = (B_1B_2)$. Therefore A contains an element α such that $Q_1^{\alpha^{\mathcal{B}}} = Q_2$. ■

We conclude that we can assume that $Q_1 = Q_2$.

4.3 Step 3

Let us now assume that $Q_1 = Q_2$. We intend to find $\gamma \in A$ such that $\mathring{Q}^{\gamma} = \hat{Q}$.

Let $\hat{x}$ and $\mathring{x}$ denote the generators of $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$, respectively. We may assume that $\hat{x} \restriction B_1 = \mathring{x} \restriction B_1$.

Lemma 10. *There exists $\gamma \in A$ such that $\mathring{x}^\gamma = \hat{x}$.*

Proof. Let us assume first that Γ_0 is connected. In this case there is only one equivalence class of size 8. It is clear by Lemma 6 (b) that $\mathring{x} = \hat{x}$.

Let us assume that Γ_0 is not connected. In this case there are at least two equivalence classes which we denote by $\mathscr{C}_1, \dots, \mathscr{C}_n$. The permutations $\hat{x}$ and $\mathring{x}$ are elements of the base group of $\hat{\mathbb{Z}}_p \wr Sym(8)$ and hence they can be considered as functions on $\mathcal{B}$. By Lemma 6 (b) $\mathring{x}$ is constant on every equivalence class and we may assume that $\hat{x}(q,u) = (q, u+1)$ for every $(q,u) \in Q \times \mathbb{Z}_p$. We may also assume that $B_1 \in \mathscr{C}_1$.

For every $1 \leq m \leq n$ there exists $\mathring{q}_m \in \mathring{Q}$ such that $\mathring{q}_m(\mathscr{C}_1) = \mathscr{C}_m$ and for every $\mathring{q}_m \in \mathring{Q}$ there exists $\hat{q}_m \in \hat{Q}$ such that $\mathring{q}_m^{\mathcal{B}} = \hat{q}_m^{\mathcal{B}}$. Let γ be defined as follows:

$$\gamma \restriction \cup\mathscr{C}_1 = id$$
$$\gamma \restriction \cup\mathscr{C}_m = \mathring{q}_m \hat{q}_m^{-1} \quad \text{for } 2 \leq m \leq n.$$

Let $(b,v) \in \mathring{q}_m(B_e)$ with $B_e \in \mathscr{C}_1$ and we denote $\mathring{q}_m^{-1}(b,v)$ by (a,u). Since $\mathring{x}$ is constant on $\mathscr{C}_m$ we have $\mathring{x}^s(b,v) = (b, v + c_m s)$ for some c_m which only depends on $\mathscr{C}_m$. Thus $\mathring{q}_m(a, u+s) = (b, v + c_m s)$ since $\mathring{x}$ and $\mathring{q}_m$ commute and $\mathring{x} \restriction B_e = \hat{x} \restriction B_e$. Therefore for every $w \in \mathbb{Z}_p$ we have

$$\gamma(b,w) = \mathring{q}_m(a,w) = \mathring{q}_m(a, u + (w-u)) = (b, v + c_m(w-u))$$

for every $(b,w) \in \mathring{q}_m(B_e)$. It is easy to verify that $\gamma^{-1}(b,w) = (b, \frac{w-v+uc_m}{c_m})$ for every $w \in \mathbb{Z}_p$ which gives

$$\gamma^{-1}\mathring{x}\gamma(b,w) = \gamma^{-1}\mathring{x}(b, wc_m + v - uc_m) = \gamma^{-1}(b, wc_m + v - uc_m + c_m) = (b, w+1).$$

It remains to show that $\gamma \in A$. Let y and z be two points of $Q \times \mathbb{Z}_p$.

If y and z are in the same equivalence class $\mathscr{C}_m$, then either γ is defined on y and z by $\mathring{q}_m \hat{q}_m^{-1}$ which is the element of the group $\langle \mathring{G}, \hat{G} \rangle \leq A$ or $\gamma(y) = y$ and $\gamma(z) = z$.

We denote by B_y and B_z the elements of $\mathcal{B}$ containing y and z, respectively. If y and z are not in the same equivalence class, then $B_y \sim B_z$. The definition of γ shows that $\gamma^{\mathcal{B}} = id$. Using Lemma 6 (c) we get that $\gamma \restriction B_y \cup B_z$ is an automorphism of the induced subgraph of Γ on the set $B_y \cup B_z$, which proves that $\gamma \in A$, finishing the proof of Lemma 10. ■

Using Lemma 10 we may assume that $\mathring{x} = \hat{x}$. Since $\mathring{x}$ and $\mathring{q}$ commute we have $\mathring{Q} \times \mathring{\mathbb{Z}}_p \leq \hat{\mathbb{Z}}_p \wr Sym(8)$. Now we can apply Lemma 7 which gives $(\mathring{q})_b = id$ for every $\mathring{q} \in \mathring{Q}$. This proves that $\mathring{Q} = \hat{Q}$ since $Q_1 = Q_2$. Therefore $\mathring{G} = \hat{G}$, finishing the proof of Proposition 2.

Our method also gives the analogous result for $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, what also follows from the theorem of Dobson and Spiga [3].

Proposition 3. *For every prime $p > 8$, the group $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ is a DCI-group.*

In order to prove Proposition 3 we will mofidy the proof of Proposition 2. Let Γ be a Cayley graph of $G = \mathbb{Z}_2^3 \times \mathbb{Z}_p$ and let $A = Aut(\Gamma)$. Let $\mathring{G} = \mathring{\mathbb{Z}}_2^3 \times \mathring{\mathbb{Z}}_p$ be a regular subgroup of A isomorphic to $\mathbb{Z}_2^3 \times \mathbb{Z}_p$. It is enough to prove that there exists $\alpha \in A$ such that $\mathring{G}^\alpha = (\mathring{\mathbb{Z}}_2^3 \times \mathring{\mathbb{Z}}_p)^\alpha = \hat{\mathbb{Z}}_2^3 \times \hat{\mathbb{Z}}_p = \hat{G}$.

It is easy to verify that the argument of the first step in subsection 4.1 only uses the fact that $p > 8$. Therefore there exists a P-invariant partition $\mathcal{D} = \{D_1, D_2, \ldots, D_8\}$, where P is a Sylow p-subgroup of $Sym(8p)$ containing $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$. We denote be A_1 and A_2 the regular action on $\mathcal{D}$ induced by $\hat{\mathbb{Z}}_2^3$ and $\mathring{\mathbb{Z}}_2^3$, respectively.

Let us assume that $A_1 \neq A_2$. We will repeat the argument of Step 2. Similarly to the definition of Γ_1 one can define a colored graph Γ_1' on $\mathcal{D}$. Since $\mathbb{Z}_2^3$ is also a $DCI^{(2)}$-group there exists $\beta' \in Aut(\Gamma_1')$ such that $A_2^{\beta'} = A_1$.

One can also define the graph Γ_0' using the relation $\equiv$ and similarly to Lemma 7 one can prove that if Γ_0' is connected, then there exists $\beta \in A$ such that $\beta^{\mathcal{D}} = \beta'$.

If Γ_0' is the empty graph, then every automorphism of Γ_1' lifts to an automorphism of Γ.

Similarly to Observation 2 the automorphism group A contains δ_1, δ_2, δ_3 such that

$$\delta_1^{\mathcal{D}} = (D_1D_2)(D_3D_4),\ \delta_2^{\mathcal{D}} = (D_1D_2)(D_5D_6),\ \delta_3^{\mathcal{D}} = (D_1D_2)(D_7D_8).$$

since $\langle A_1, A_2\rangle = \langle Q_1, Q_2\rangle$ by Lemma 5.

It is straightforward to check that Lemma 8 and Lemma 9 only uses the existence of the involutions $\beta_1, \beta_2, \beta_3$ so the argument can be repeated using δ_1, δ_2 and δ_3. Therefore we may assume that $A_1 = A_2$.

Finally, the proof of Lemma 10 can also be repeated for $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ which gives that the generators of $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$ coincide. Since $A_1 = A_2$ we have $\hat{G} = \mathring{G}$, finishing the proof of Proposition 3. ■

It is straigtforward to check that the proof of Proposition 2 and Proposition 3 only uses the fact that $p > 8$ in the first step of the argument. We can formulate this fact in Proposition 4.

Proposition 4. *Let Γ be a Cayley graph of $G = Q \times \mathbb{Z}_p$ or $G = \mathbb{Z}_2^3 \times \mathbb{Z}_p$, where p is an odd prime and let $\mathring{G} = \mathring{Q} \times \mathring{\mathbb{Z}}_p$ or $\mathring{G} = \mathring{\mathbb{Z}}_2^3 \times \mathring{\mathbb{Z}}_p$ be a regular subgroup of $Aut(\Gamma)$ isomorphic to G. Let us assume that there exists a $\langle \hat{G}, \mathring{G}\rangle$-invariant partition $\mathcal{B} = \{B_1, B_2, \ldots, B_8\}$ of $V(\Gamma)$, where $|B_i| = p$ for every $i = \{1, \ldots, 8\}$. In addition, we assume that $\mathring{\mathbb{Z}}_p$ is a subgroup of the base group of $\hat{\mathbb{Z}}_p wr Sym(\mathcal{B})$. Then there is an automorphism α of the graph Γ such that $\hat{G}^\alpha = \mathring{G}$.*

5 Main result for $p = 5$ and 7

In this section we will prove that $Q \times \mathbb{Z}_5$, $Q \times \mathbb{Z}_7$, $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ are CI-groups.

The whole section is based on the paper [5], so we will only modify the proof of Lemma 5.4 of [5].

Proposition 5. *Every Cayley graph of $Q \times \mathbb{Z}_5$, $Q \times \mathbb{Z}_7$, $\mathbb{Z}_2^3 \times \mathbb{Z}_5$ and $\mathbb{Z}_2^3 \times \mathbb{Z}_7$ is a CI-graph.*

We denote by R one of the groups Q and $\mathbb{Z}_2^3$. Let Γ be a Cayley graph of one of these groups, $A = Aut(\Gamma)$ and P a Sylow p-subgroup of A for $p = 5, 7$, respectively. Let us assume that A contains two copies of regular subgroups which we denote by $\hat{G} = \hat{R} \times \hat{\mathbb{Z}}_p$ and $\mathring{G} = \mathring{R} \times \mathring{\mathbb{Z}}_p$. We can assume that Γ is neither the empty nor the complete graph and both $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$ are contained in P.

It was proved in [5] that the action of A on the points of graph Γ cannot be primitive so there is a nontrivial A-invariant partition $\mathcal{B} = \{B_0, B_1, \ldots, B_{t-1}\}$ of $V(\Gamma) = G$. The elements of the partition $\mathcal{B}$ have the same cardinality since the action of A is transitive on $\mathcal{B}$ so $|B_i| < p^2$ for every $i = 0, 1, \ldots, t-1$. The partition $\mathcal{B}$ is P-invariant so P acts on $\mathcal{B}$. Since P is a p-group, the length of every orbit of P is a power of p.

If the length of every orbit of P on $V(\Gamma)$ is p, then it is clear from Proposition 4 that Γ is a CI-graph. Therefore P has an orbit $\Lambda \subset G$ such that $|\Lambda| = p^2$ since $p^3 > |G|$ and the remaining orbits of P have length p since $2p^2 > 8p$.

Let $\mathscr{C} = \{C_0, C_1, \ldots, C_{s-1}\}$ be an orbit of P on $\mathcal{B}$ such that $\Lambda \subseteq \cup_{i=0}^{s-1} C_i$. We may assume that $B_i = C_i$ for $i = 0, 1, \ldots, s-1$. It is clear that s is a power of p. If $s \geq p^2$, then $\left|\cup_{i=0}^{s-1} C_i\right| \geq 2p^2 > 8p$ which is a contradiction. It follows that $1 < s < p^2$ which implies $s = p$.

For every $i < s$ and every $x \in P$ the following eqalities hold for some $j < s$

$$(B_i \cap \Lambda)^x = {B_i}^x \cap \Lambda^x = B_j \cap \Lambda.$$

This implies that

$$|B_0 \cap \Lambda| = |B_i \cap \Lambda|$$

for every $0 \leq i < s$. Therefore

$$p^2 = |\Lambda| = \left|\cup_{i=0}^{s-1} (B_i \cap \Lambda)\right| = s\,|B_0 \cap \Lambda| = p\,|B_0 \cap \Lambda|\,.$$

This gives $|B_0 \cap \Lambda| = p$ so $|B_0|$ can only be p or 8 since $|B_0|\,t = 8p$ and both $|B_0|$ and t are at least p.

If $|B_0| = p$, then Λ is the union of p elements of the A-invariant partition $\mathcal{B}$ and every orbit Λ' of P is an element of the partition $\mathcal{B}$ if $\Lambda' \neq \Lambda$. For every orbit $\Lambda' \neq \Lambda$ of P and for every $y \in \hat{\mathbb{Z}}_p \cup \mathring{\mathbb{Z}}_p$ we have $y(\Lambda') = \Lambda'$. By Proposition 4 we may assume that there exists an element x' in $\hat{\mathbb{Z}}_p \cup \mathring{\mathbb{Z}}_p$ such that $x'(B_0) \neq B_0$ and clearly $x'(B_7) = B_7$ for every $x' \in \hat{\mathbb{Z}}_p \cup \mathring{\mathbb{Z}}_p$. Since both $\hat{G}$ and $\mathring{G}$ are regular there exists $a \in C_A(x')$ such that $a(B_0) = B_7$, which contradicts the fact that $\mathcal{B}$ is A-invariant and B_7 is an orbit of P.

Let us assume that $|B_0| = 8$ and let $\hat{x}$ and $\mathring{x}$ generate $\hat{\mathbb{Z}}_p$ and $\mathring{\mathbb{Z}}_p$, respectively. Since $\hat{G}$ and $\mathring{G}$ are regular we have that neither $\hat{x}^{\mathcal{B}}$ nor $\mathring{x}^{\mathcal{B}}$ is the identity, while for every $r \in \hat{R} \cup \mathring{R}$ we have $r^{\mathcal{B}} = id$. Since $\hat{x}$ and $\mathring{x}$ are in the same Sylow p-subgroup of P we may assume that $\hat{x}(B_i) = \mathring{x}(B_i) = B_{i+1}$ for $i = 0, 1, \ldots, p-1$, where the indices are taken modulo p. By Proposition 4 we may also assume that $\hat{x} \neq \mathring{x}$.

For every m there exists an l such that the action of $\hat{x}^l \mathring{x}^{-l}$ is nontrivial on B_m since $\hat{x} \neq \mathring{x}$. Therefore $A_{B_m} \restriction B_m$ contains a regular subgroup and a cycle of length p such that $p > \frac{|B_0|}{2}$. A theorem of Jordan says that such a permutation group is 2-transitive and hence the induced subgraph by B_m of Γ is the complete or the empty graph for every m.

Lemma 11. *$B_m \sim B_n$ for $0 \leq m \neq n \leq p-1$.*

Proof. There exists a unique element $\hat{g} \in \hat{\mathbb{Z}}_p \leq P$ such that $\hat{g}(B_m) = B_n$. We also have a unique element $\mathring{g} \in \mathring{\mathbb{Z}}_p \leq P$ with $\hat{g}^{\mathcal{B}} = \mathring{g}^{\mathcal{B}}$. Since $\mathbb{Z}_p$ is cyclic and $\hat{x} \neq \mathring{x}$ we have $\hat{g} \neq \mathring{g}$. Moreover, we may also assume that $\hat{g} \restriction B_m \neq \mathring{g} \restriction B_m$ since $\hat{g} \neq \mathring{g}$ and the induced subgraphs of Γ by $B_{m+c} \cup B_{n+c}$ are all isomorphic, where both $m+c$ and $n+c$ are taken modulo p.

Clearly, $\tilde{g} = \mathring{g}\hat{g}^{-1}$ is cycle of length p on B_n. The points of $V(\Gamma) \setminus \Lambda$ are contained in P-orbits of length p so and $\tilde{g}$ fixes every point of the set $B_m \cup B_n \setminus \Lambda$ since $\tilde{g}^{\mathcal{B}} = id$.

Let $u \in B_m \setminus \Lambda$. It is enough to show that if u is connected to some $v \in B_n$, then u is connected to every point of B_n. We will prove that A is transitive on the following pairs: $\{(u,w) \mid w \in B_n\}$.

A is transitive on $\{(u,w) \mid w \in B_n \cap supp(\tilde{g})\} = \{(u,w) \mid w \in B_n \cap \Lambda\}$ since $\tilde{g}$ fixes u. Therefore we may assume that $v \in B_n \setminus \Lambda$ and we only have to find an element $a \in A$ such that $a(u) = u$ and $a(v) \in B_n \cap \Lambda$.

The restriction of $\tilde{g}$ to B_n is a cycle of length p which does not commute with $\mathring{r} \restriction B_n$, where $\mathring{r}$ is an involution of $\mathring{R}$. Since $\mathring{r}$ and $\mathring{g}$ commute we have that there is a $u' \in B_m$ such that $\mathring{r}\hat{g}(u') \neq \hat{g}\mathring{r}(u')$. Since the action of $\hat{R}$ is transitive on B_m there exists $\hat{r} \in \hat{R}$ such that $\hat{r}(u) = u'$. Then

$$(\mathring{r}\hat{r})\,\hat{g}(u) = \mathring{r}\hat{g}\hat{r}(u) = \mathring{r}\hat{g}(u') \neq \hat{g}\mathring{r}(u') = \hat{g}\,(\mathring{r}\hat{r})\,(u)$$

so there exists $a' \in A$ such that

$$a'\hat{g}(u) \neq \hat{g}a'(u). \qquad (1)$$

Let us assume that $v = \hat{g}(u)$. Then the inequality 1 gives $a'(v) \neq \hat{g}a'(u)$. Since $\hat{R} \restriction B_m$ is regular on B_m there exists $\hat{s} \in \hat{R}$ such that $\hat{s}(u) = a'(u)$ and since $\hat{s}$ and $\hat{g}$ commute we have $\hat{s}(v) = \hat{s}\hat{g}(u) = \hat{g}\hat{s}(u) = \hat{g}a'(u)$. Therefore $\hat{s}(v) \neq a'(v)$ and hence $\hat{s}^{-1}a'$ fixes u and $\hat{s}^{-1}a'(v) \neq v$ so we may assume that $v \neq \hat{g}(u)$.

If $p = 7$, then $v \in B_n \cap \Lambda$.

If $p = 5$, then there exists $\hat{t} \in \hat{R}$ such that $\hat{t}(u) \in B_m \setminus \Lambda = B_m \setminus supp(\tilde{g})$ while $\hat{t}(v) \in B_n \cap \Lambda \subset supp(\tilde{g})$ since both $\hat{R} \restriction B_m$ and $\hat{R} \restriction B_n$ are regular and $\gcd(8,5) = 1$. The permutations $\hat{t}^{-1}\tilde{g}^l\hat{t}$ fix the point u for every $0 \leq l \leq 4$ and $\hat{t}^{-1}\tilde{g}^{l_1}\hat{t}(y) \neq \hat{t}^{-1}\tilde{g}^{l_2}\hat{t}(y)$ if $l_1 \not\equiv l_2 \ (mod\, p)$. At least one of the the four elements $\hat{t}^{-1}\tilde{g}\hat{t}, \hat{t}^{-1}\tilde{g}^2\hat{t}, \hat{t}^{-1}\tilde{g}^3\hat{t}, \hat{t}^{-1}\tilde{g}^4\hat{t}$ of A fixes u and maps v to an element of $B_n \cap fix(\tilde{g}) = B_n \cap \Lambda$ since $|B_n \setminus supp(\tilde{g})| = 3$, finishing the proof of the fact that $B_m \sim B_n$ for $0 \leq m \neq n \leq 7$. ■

Every permutation of $V(\Gamma)$ which fixes setwise B_m for every m is an automorphism of Γ so there is an $a \in A$ such that $\mathring{x}^a = \hat{x}$. Applying Proposition 4 we get that there exists $\alpha \in A$ such that $\left(\hat{R} \times \hat{\mathbb{Z}}_p\right)^\alpha = \mathring{R} \times \mathring{\mathbb{Z}}_p$, finishing the proof of Proposition 5.

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