AAL, Ex. 6.

1. (a) Prove that A_n is generated by all of its 3-cycles.
   (b) Prove that S_n is generated by $(1,2)$ and $(1,2,\ldots,n)$
   (c*) Prove that if $m < n$, then S_n is generated by $(1,2,\ldots,m)$ and $(1,2,\ldots,n)$ unless n and m are odd. In this case these two elements generate A_n.

2. Prove that $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL(2,\mathbb{Z})$

3. Let G be an abelian group.
   (a) Prove: a subgroup $H \leq G$ is maximal if and only if $|G : H|$ is a prime number.
   (b) Suppose G is multiplicatively written. Prove: $\phi(G) = \cap G^p$ with p ranging over the set of prime numbers.

4. Let G be a finite group. Prove: G is nilpotent if and only if each subgroup of G is subnormal (A subgroup H is subnormal iff. there exists k and $H \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \ldots \trianglelefteq N_k = G$).

5. Let G be a finite p-group and $\phi(G)$ the Frattini subgroup of G. We know that $G/\phi(G)$ is isomorphic to (the vector space) C_p^d for some positive integer d.
   (a) Verify that there is a natural homomorphism from $Aut(G)$ to $Aut(G/\phi(G))$.
   (b) Prove that the order of the kernel of this homomorphism divides $|\phi(G)|^d$.

6. What is $\phi(\mathbb{Z})$?

7. What is $\phi(GL(3,\mathbb{Q}))$?